

Isomorphisms of Cayley graphs on nilpotent groups

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ABSTRACT. Let S be a finite generating set of a torsion-free, nilpotent group G . We show that every automorphism of the Cayley graph $\text{Cay}(G; S)$ is affine. (That is, every automorphism of the graph is obtained by composing a group automorphism with multiplication by an element of the group.) More generally, we show that if $\text{Cay}(G_1; S_1)$ and $\text{Cay}(G_2; S_2)$ are connected Cayley graphs of finite valency on two nilpotent groups G_1 and G_2 , then every isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$ factors through to a well-defined affine map from G_1/N_1 to G_2/N_2 , where N_i is the torsion subgroup of G_i . For the special case where the groups are abelian, these results were previously proved by A. A. Ryabchenko and C. Löh, respectively.

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1. Introduction

It is easy to construct examples of non-isomorphic groups that have isomorphic Cayley graphs, even if the Cayley graphs are required to be connected and have finite valency. We show that this is not possible when the groups are torsion-free and nilpotent:

(see
note
A.1)

Theorem 1.1. *Suppose G_1 and G_2 are torsion-free, nilpotent groups. If G_1 has a connected Cayley graph of finite valency that is isomorphic to a Cayley graph on G_2 , then $G_1 \cong G_2$.*

In fact, the next theorem establishes the stronger conclusion that every

(see
note
A.2)

2010 *Mathematics Subject Classification.* 05C25, 20F18, 05C63, 20F65.

Key words and phrases. Cayley graph, nilpotent group, isomorphism, Cayley isomorphism property.

isomorphism of the Cayley graphs is obtained from an isomorphism of the groups.

Definition 1.2. Suppose $\varphi: G_1 \rightarrow G_2$, where G_1 and G_2 are groups. We say that φ is an *affine bijection* if it is the composition of a group isomorphism and a translation. That is, there exist a group isomorphism $\alpha: G_1 \rightarrow G_2$ and $h \in G_2$, such that $\varphi(x) = h \cdot \alpha(x)$, for all $x \in G_1$.

Theorem 1.3. *Assume*

- G_1 and G_2 are torsion-free, nilpotent groups, and
- S_i is a finite, symmetric generating set of G_i , for $i = 1, 2$.

Then every isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$ is an affine bijection.

Remark 1.4. In the special case where G_1 and G_2 are abelian, Theorem 1.3 was proved by A. A. Ryabchenko [12].

(see
note
A.3)

Definition 1.5. [7, §6.4] Let G be a group. A Cayley graph $\text{Cay}(G; S)$ is said to be *normal* if the left-regular representation of G is a normal subgroup of $\text{Aut}(\text{Cay}(G; S))$ or, equivalently [2, Lem. 2.2(b)], if every automorphism of $\text{Cay}(G; S)$ is an affine bijection.

Remark 1.6. It is easy to see that the left-regular representation of G is a subgroup of the automorphism group of every Cayley graph on G . 1.5 requires this subgroup to be normal.

With this terminology, the special case of Theorem 1.3 in which $G_1 = G_2$ has the following known result as an immediate consequence.

Corollary 1.7 (Möller-Seifter [9, Thm. 4.1(1)]). *If G is a torsion-free, nilpotent group, then every connected Cayley graph of finite valency on G is normal.*

In the statement of Theorem 1.3, the word “nilpotent” cannot be replaced with “solvable” (or even “polycyclic”):

Example 1.8. Let G be the unique nonabelian semidirect product of the form $\mathbb{Z} \rtimes \mathbb{Z}$. More precisely,

$$G = \langle a, b \mid b^{-1}ab = a^{-1} \rangle = \langle a \rangle \rtimes \langle b \rangle.$$

(In other words, G is the fundamental group of the Klein bottle.) Then G is obviously polycyclic (so it is solvable), but it is not difficult to see that $\text{Cay}(G; \{a^{\pm 1}, b^{\pm 1}\})$ is not normal. (Namely, the map $\varphi(a^i b^j) = b^i a^j$ is a graph automorphism that is not an affine bijection.)

(see
note
A.4)

If G is not torsion-free, then the conclusion of Corollary 1.7 fails:

Proposition 1.9. *Let G be a finitely generated, infinite group. If G is not torsion-free, then G has a connected Cayley graph of finite valency that is not normal.*

However, the next theorem shows that if the torsion-free hypothesis is removed from Theorem 1.3, then the conclusion still holds modulo the elements of finite order.

Definition 1.10 ([6, 1.2.13, p. 11]). Suppose G is a finitely generated, nilpotent group. The set of all elements of finite order in G is called the *torsion subgroup* of G . (This is a finite, normal subgroup of G .)

Theorem 1.11. *Assume*

- S_i is a symmetric, finite generating set of the nilpotent group G_i , for $i = 1, 2$,
- φ is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$, and
- N_i is the torsion subgroup of G_i , for $i = 1, 2$.

Then φ induces a well-defined affine bijection $\bar{\varphi}: G_1/N_1 \rightarrow G_2/N_2$.

Corollary 1.12. *For $i = 1, 2$, assume N_i is the torsion subgroup of the finitely generated, nilpotent group G_i . Then there is a connected Cayley graph of finite valency on G_1 that is isomorphic to a Cayley graph on G_2 if and only if $G_1/N_1 \cong G_2/N_2$ and $|N_1| = |N_2|$.*

Corollary 1.13. *If $\text{Cay}(G; S)$ is any Cayley graph of finite valency on a torsion-free, nilpotent group G , then the left-regular representation of G is the only nilpotent subgroup of $\text{Aut}(\text{Cay}(G; S))$ that acts sharply transitively on the vertices of the Cayley graph.*

Remarks 1.14.

- (1) In the special case where G_1 and G_2 are abelian, Theorem 1.11 and Corollary 1.12 were proved by C. Löh [8].
- (2) Theorem 1.3 is the special case of Theorem 1.11 in which the torsion subgroups N_1 and N_2 are trivial.
- (3) Although Theorems 1.3 and 1.11 are stated only for graphs, they obviously remain true in the setting of Cayley digraphs. This is because any isomorphism of digraphs is also an isomorphism of the underlying graphs.
- (4) Some non-nilpotent groups have some Cayley graphs that are isomorphic to Cayley graphs on nilpotent groups—or even abelian groups. (For example, the Cayley graph in Example 1.8 is isomorphic to $\text{Cay}(\mathbb{Z} \times \mathbb{Z}, \{(\pm 1, 0), (0, \pm 1)\})$.) Theorem 1.11 implies that any such group must have a subgroup of finite index that is nilpotent, but this fact is well known to be a consequence of Gromov’s famous theorem that groups of polynomial growth are virtually nilpotent [3]. Indeed, in order to conclude from Gromov’s Theorem that G has a nilpotent subgroup of finite index, it suffices to know that G has a connected Cayley graph of finite valency that is *quasi-isometric* (not necessarily isomorphic) to a Cayley graph on a nilpotent group.

Theorem 1.3 is proved in Section 3, and this result is used to prove Theorem 1.11 (and its corollaries) in Section 4. (The arguments are based

on techniques of A.A. Ryabchenko [12] and C. Löh [8].) Proposition 1.9 is proved in Section 5.

Acknowledgments. This work was partially supported by Australian Research Council grant DE130101001 and a research grant from the Natural Sciences and Engineering Research Council of Canada.

2. Preliminaries

The following result is the special case of Theorem 1.3 in which G_1 and G_2 are abelian. (Although not stated in exactly this form in [12], the result follows from the proof that is given there and is reproduced in [10, Thm. 5.3]). This case is not covered by the proof in Section 3.

Proposition 2.1 (Ryabchenko [12, Thm. 2]). *Assume*

- G_1 and G_2 are torsion-free, abelian groups,
- S_i is a symmetric, finite generating set of G_i , for $i = 1, 2$, and
- φ is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$.

(see
note
A.5)

Then φ is an affine bijection.

As in [8], we use geometric terminology, such as geodesics and convexity, instead of presenting our arguments in group-theoretic language.

Definition 2.2. Let S be a symmetric, finite generating set of a group G .

- For $g, h \in G$, the distance from g to h in the Cayley graph $\text{Cay}(G; S)$ is denoted $\text{dist}_S(g, h)$.
- A finite sequence $[g_i]_{i=m}^n$ of elements of G is a *geodesic segment* from g_m to g_n in $\text{Cay}(G; S)$ if $\text{dist}_S(g_i, g_j) = |i - j|$ for $m \leq i, j \leq n$.
- A bi-infinite sequence $[g_i]_{i=-\infty}^{\infty}$ of elements of G is a *geodesic line* in $\text{Cay}(G; S)$ if $\text{dist}_S(g_i, g_j) = |i - j|$ for all $i, j \in \mathbb{Z}$.
- A geodesic line $[g_i]_{i=-\infty}^{\infty}$ in $\text{Cay}(G; S)$ is *convex* if $[g_i, g_{i+1}, \dots, g_j]$ is the only path of length $j - i$ from g_i to g_j , for all $i, j \in \mathbb{Z}$ (with $i < j$).
- A geodesic line $[g_i]_{i=-\infty}^{\infty}$ in $\text{Cay}(G; S)$ is *homogeneous* if there exists $\varphi \in \text{Aut}(\text{Cay}(G; S))$, such that $\varphi(g_i) = g_{i+1}$ for all i .
- $\text{Aut}_e(\text{Cay}(G; S)) = \{ \varphi \in \text{Aut}(\text{Cay}(G; S)) \mid \varphi(e) = e \}$.
- Each oriented edge of $\text{Cay}(G; S)$ has a natural label, which is an element of S . Namely, each edge of the form $g \xrightarrow{\quad} gs$ is labelled s . (Note that the same edge with the opposite orientation is labelled s^{-1} .) Each edge in a geodesic segment (or geodesic line) comes with a natural orientation, and therefore has a label.

Lemma 2.3. *For $i = 1, 2$, assume*

- S_i is a symmetric, finite generating set of a group G_i ,
- φ_i is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$, such that $\varphi_i(e) = e$,
- $g_i \in G_i$,

- $S_i^* = \{ \rho(g_i) \mid \rho \in \text{Aut}_e(\text{Cay}(G_i; S_i)) \}$, and
- $G_i^* = \langle S_i^* \rangle$.

If $\varphi_1(g_1) = g_2$, then the restriction of φ_2 to G_1^* is an isomorphism from $\text{Cay}(G_1^*; S_1^* \cup (S_1^*)^{-1})$ to $\text{Cay}(G_2^*; S_2^* \cup (S_2^*)^{-1})$.

Proof. For convenience, let $A_i = \text{Aut}(\text{Cay}(G_i; S_i))$, $A_i^e = \text{Aut}_e(\text{Cay}(G_i; S_i))$, and $\Gamma_i = \text{Cay}(G_i; S_i^* \cup (S_i^*)^{-1})$. For $\rho \in A_i$ and $g \in G_i$, define $\rho_g \in A_i^e$ by $\rho_g(x) = \rho(g)^{-1} \rho(gx)$. Then, since S_i^* is A_i^e -invariant, we have

$$\rho(gS_i^*) = \rho(g) \rho_g(S_i^*) = \rho(g)S_i^*,$$

so ρ is an automorphism of Γ_i . Since A_i^e is transitive on S_i^* , and the left-regular representation of G_i is transitive on G_i , this implies that the set of edges of Γ_i is the A_i -orbit of the edge $e = g_i$.

Since φ_1 is a graph isomorphism, it maps the A_1 -orbit of g_1 to the A_2 -orbit of $\varphi_1(g_1) = g_2$. So φ_1 is an isomorphism from Γ_1 to Γ_2 . Since the composition $\varphi_2 \circ \varphi_1^{-1}$ is in A_2 , and is therefore an automorphism of Γ_2 , we conclude that φ_2 is an isomorphism from Γ_1 to Γ_2 . Since $\text{Cay}(G_i^*; S_i^* \cup (S_i^*)^{-1})$ is the component of Γ_i that contains e , and $\varphi_2(e) = e$, the desired conclusion follows. \square

Lemma 2.4 ([8, Prop. 2.5(3)]). *Let $s \in S$ be the label of some edge of a convex geodesic line in $\text{Cay}(G; S)$. If $s \in Z(G)$, then every edge of the geodesic line is labelled s .*

Proof. Suppose $g_i = g_{i+1}$ is labelled s . Let t be the label of $g_{i+1} = g_{i+2}$. Then $g_{i+2} = g_i s t = (g_i t) s$, so $[g_i, g_i t, g_{i+2}]$ is a path of length 2 from g_i to g_{i+2} . Therefore, convexity implies $[g_i, g_i t, g_{i+2}] = [g_i, g_{i+1}, g_{i+2}]$, so $g_i t = g_{i+1} = g_i s$, so $t = s$. This means the label of $g_{i+1} = g_{i+2}$ is s . By induction, we see that every edge is labelled s . \square

In the remainder of this section, we recall some basic facts about nilpotent groups.

Definition 2.5 ([6, p. 38] or [1, Notn. 3.4]). For a subgroup H of a group G , we let

$$\sqrt{H} = \{ g \in G \mid g^k \in H \text{ for some } k \in \mathbb{Z}^+ \}.$$

This is called the *isolator* of H in G .

Any finitely generated, abelian group A is isomorphic to $\mathbb{Z}^r \times F$, for some $r \in \mathbb{Z}^{\geq 0}$ and finite, abelian group F . The number r is called the *rank* of A , and is denoted $\text{rank } A$. The following definition generalizes this notion from abelian groups to nilpotent groups.

Definition 2.6 ([6, 1.3.3 and p. 85 (1)]). Assume G is a nilpotent group. Then G is solvable, which means there is a series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{r-1} \triangleleft G_r = G,$$

of subgroups of G , such that each quotient G_i/G_{i-1} is abelian. If G is finitely generated, then the *Hirsch rank* of G is the sum of the ranks of these (finitely generated) abelian groups. That is,

$$\text{rank } G = \sum_{i=1}^r \text{rank}(G_i/G_{i-1}).$$

It is not difficult to see that this is independent of the choice of the subgroups G_1, \dots, G_{r-1} .

Lemma 2.7. *Assume G is a finitely generated, nilpotent group, H is a subgroup of G , and S is a symmetric, finite generating set of G . Then:*

- (1) [6, 1.2.16, p. 11] H is finitely generated.
- (2) [6, 2.3.1(ii), p. 39] \sqrt{H} is a subgroup of G that contains H , and $|\sqrt{H} : H| < \infty$.
- (3) If $N \trianglelefteq G$, then $\sqrt{N} \trianglelefteq G$ and G/\sqrt{N} is torsion-free.
- (4) [6, 2.3.8(ii), p. 42] If G is torsion-free, then $\sqrt{Z(G)} = Z(G)$.
- (5) [6, 2.3.9(iv), p. 43] $[\sqrt{H}, \sqrt{H}] \subseteq \sqrt{[H, H]}$.
- (6) If N is a normal subgroup of G , then $\text{rank } G = \text{rank } N + \text{rank}(G/N)$. Therefore, $\text{rank}(G/N) \leq \text{rank } G$, with equality if and only if N is finite.
- (7) (cf. [4, Lem. 2.6, p. 9]) We have $\text{rank } H \leq \text{rank } G$, with equality if and only if $|G : H| < \infty$.
- (8) [11, 5.2.1, p. 129] If N is a nontrivial normal subgroup of G , then $N \cap Z(G)$ is nontrivial.
- (9) (cf. [6, 2.3.8(i), p. 42]) If G is torsion-free, then the elements of $Z(G)$ are the only elements of G that have only finitely many conjugates.
- (10) (cf. [6, 2.1.2, p. 30]) Assume G is torsion-free, and $a, b, g \in G$. If

$$\sup_{k \in \mathbb{Z}^+} \text{dist}_S(a^k, gb^k) < \infty,$$

then $b = g^{-1}ag$.

- (11) [1, Lem. 3.5(i,iii)] For $g \in G$, we have $g \in \sqrt{[G, G]}$ if and only if $\text{dist}_S(e, g^k)/k \rightarrow 0$ as $k \rightarrow \infty$.

Remark 2.8. Lemma 2.7(5) corrects a typographical error. It is stated in [6, 2.3.9(iv), p. 43] that equality holds, but a counterexample to this is provided by any finite-index subgroup G of the discrete Heisenberg group, such that $[G, G]$ is a proper subgroup of $Z(G)$: letting $H = G$, we have

$$[\sqrt{G}, \sqrt{G}] = [G, G] \neq Z(G) = \sqrt{[G, G]}.$$

Definition 2.9. A group G is *bi-orderable* if it has a total order \prec that is invariant under both left-translations and right-translations. (That is, $x \prec y \Rightarrow axb \prec ayb$ for all $x, y, a, b \in G$.)

Lemma 2.10 ([5, Cor. 3.3.2, p. 57]). *Every torsion-free, nilpotent group is bi-orderable.*

Lemma 2.11 (cf. [12, 1st paragraph of §4] or [8, Prop. 2.9(1)]). *If S is a finite generating set of a nontrivial, bi-orderable group G , then there exists $s \in S$, such that $[s^i]_{i=-\infty}^{\infty}$ is a convex geodesic line in $\text{Cay}(G; S \cup S^{-1})$.*

Proof. Let \prec be a total order on G that is invariant under both left-translations and right-translations. Since the set $S \cup S^{-1}$ is finite, it has a maximal element s under this order. We may assume $s \in S$, by replacing \prec with the order \prec' defined by $x \prec' y \Leftrightarrow x^{-1} \prec y^{-1}$, if necessary.

For $a, b, c, d \in G$ with $a \preceq b$ and $c \preceq d$, the invariance under translations implies that $ac \preceq bd$ (and equality holds iff $a = b$ and $c = d$). By induction on k , we conclude that $s_1 s_2 \cdots s_k \preceq s^k$ for all $s_1, s_2, \dots, s_k \in S \cup S^{-1}$, and that equality holds iff $s_1 = s_2 = \cdots = s_k = s$. This implies that $[s^i]_{i=-\infty}^{\infty}$ is a convex geodesic line. \square

(see
note
A.11)

(see
note
A.12)

3. Torsion-free nilpotent groups

In this section, we prove Theorem 1.3. Let φ be an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$. By composing with a left translation, we may assume $\varphi(e) = e$. (Under this assumption, we will show that φ is a group homomorphism. Since φ is bijective, it must then be a group isomorphism.) The proof is by induction on $\text{rank } G_1 + \text{rank } G_2$.

(see
note
A.13)

Notation. Let $Z_i^\dagger = Z(G_i) \cap \sqrt{[G_i, G_i]}$ for $i = 1, 2$.

Step 1. *For every $g \in G_1$ and $z \in Z_1^\dagger$, there exists $\sigma_g(z) \in G_2$, such that $\varphi(gz^k) = \varphi(g)\sigma_g(z)^k$ for all $k \in \mathbb{Z}$.*

Proof. By composing with left translations in G_1 and G_2 , we may assume $g = e$. Define S_1^* , S_2^* , G_1^* , and G_2^* as in Lemma 2.3, with $g_1 = z$ and $g_2 = \varphi(z)$. Combining Lemmas 2.10 and 2.11 yields $s \in S_2^*$, such that

(see
note
A.14)

$$[s^i]_{i=-\infty}^{\infty} \text{ is a convex geodesic line in } \text{Cay}(G_2^*; S_2^* \cup (S_2^*)^{-1}).$$

The definition of S_2^* implies there is an isomorphism ψ from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$ with $\psi(e) = e$ and $\psi(z) = s$. Since Lemma 2.3 tells us that ψ restricts to an isomorphism from $\text{Cay}(G_1^*; S_1^* \cup (S_1^*)^{-1})$ to $\text{Cay}(G_2^*; S_2^* \cup (S_2^*)^{-1})$, we know that $\psi^{-1}([s^i]_{i=-\infty}^{\infty})$ is a convex geodesic line in $\text{Cay}(G_1^*; S_1^* \cup (S_1^*)^{-1})$. From the choice of ψ , this geodesic line contains the edge $e - z$, so Lemma 2.4 tells us that this geodesic line must be $[z^i]_{i=-\infty}^{\infty}$. This means $\text{dist}_{S_1^*}(z^i, z^j) = |i - j|$ for all $i, j \in \mathbb{Z}$. We conclude from Lemma 2.7(11) that $z \notin \sqrt{[G_1^*, G_1^']}$.

(see
note
A.15)

On the other hand, since $z \in Z_1^\dagger$, we know that $z \in \sqrt{[G_1, G_1]}$. Therefore $\sqrt{[G_1^*, G_1^']} \neq \sqrt{[G_1, G_1]}$. This implies that $[G_1^*, G_1^']$ has infinite index in $[G_1, G_1]$ (cf. Lemma 2.7(2)), so G_1^* must have infinite index in G_1 (cf. Lemma 2.7(5)). Therefore, $\text{rank } G_1^* + \text{rank } G_2^* < \text{rank } G_1 + \text{rank } G_2$ (see Lemma 2.7(7)), so our induction hypothesis tells us that the restriction of φ to G_1^* is a group isomorphism onto G_2^* . Hence, $\varphi(z^k) = \varphi(z)^k$ for all k , so we may let $\sigma_g(z) = \varphi(z)$. \square

(see
note
A.16)

Step 2. We have $\varphi(xZ_1^\dagger) = \varphi(x)Z_2^\dagger$, for all $x \in G_1$.

Proof. By composing with left translations in G_1 and G_2 , we may assume $x = e$. Then, since φ^{-1} is also an isomorphism, it suffices to show $\varphi(Z_1^\dagger) \subseteq Z_2^\dagger$. Fix $z \in Z_1^\dagger$. For all $k \in \mathbb{Z}$, we have $\text{dist}_{S_1}(z^k, gz^k) = \text{dist}_{S_1}(e, g)$ (because $z \in Z(G_1)$). Since φ is a graph isomorphism, this implies $\text{dist}_{S_2}(\varphi(z)^k, \varphi(g)\sigma_g(z)^k)$ does not depend on k . So Lemma 2.7(10) tells us that $\varphi(g)^{-1}\varphi(z)\varphi(g) = \sigma_g(z)$. From the definition of $\sigma_g(z)$, we see that $\text{dist}_S(e, \sigma_g(z)) = \text{dist}_S(e, z)$, so this implies that $\varphi(g)^{-1}\varphi(z)\varphi(g)$ is in a ball of fixed radius, independent of g . Since $\varphi(g)$ is an arbitrary element of G_2 , we conclude that $\varphi(z)$ has only finitely many conjugates. Since G_2 is torsion-free nilpotent, this implies $\varphi(z) \in Z(G_2)$ (see Lemma 2.7(9)).

Also, we see from Lemma 2.7(11) that $\varphi(\sqrt{[G_1, G_1]}) = \sqrt{[G_2, G_2]}$ (since φ is a graph isomorphism). Therefore $\varphi(z) \in \sqrt{[G_2, G_2]}$. So $\varphi(z) \in Z_2^\dagger$. \square

Step 3. Completion of the proof of Theorem 1.3.

Proof. Let $\overline{G}_i = G_i/Z_i^\dagger$ for $i = 1, 2$. Note that Z_1^\dagger is finitely generated (see Lemma 2.7(1)). Therefore, by passing to a power of the graphs $\text{Cay}(G_1; S_1)$ and $\text{Cay}(G_2; S_2)$ (or, in other words, by replacing S_i with an appropriate product $(S_i \cup \{e\})(S_i \cup \{e\}) \cdots (S_i \cup \{e\})$), we may assume that $\text{Cay}(Z_1^\dagger; S_1 \cap Z_1^\dagger)$ is connected. From Step 2, we know that φ induces a well-defined isomorphism $\overline{\varphi}$ from $\text{Cay}(\overline{G}_1; S_1)$ to $\text{Cay}(\overline{G}_2; S_2)$. (see note A.17)

We may assume that G_1 and G_2 are not both abelian (otherwise, Ryabchenko's Theorem (2.1) applies), so either $[G_1, G_1]$ or $[G_2, G_2]$ is nontrivial. This implies that either Z_1^\dagger or Z_2^\dagger is nontrivial (see Lemma 2.7(8)), and therefore infinite (since G_1 and G_2 are torsion-free). Hence, we have $\text{rank } \overline{G}_1 + \text{rank } \overline{G}_2 < \text{rank } G_1 + \text{rank } G_2$ (see Lemma 2.7(6)), so, by induction on $\text{rank } G_1 + \text{rank } G_2$, we may assume that $\overline{\varphi}$ is a group isomorphism from \overline{G}_1 to \overline{G}_2 (since Lemma 2.7(4) implies that \overline{G}_1 and \overline{G}_2 are torsion free). (see note A.18)

For each $g \in G_1$ and $z \in Z_1^\dagger$, we have

$$\begin{aligned} \text{dist}_{S_2}(\sigma_e(z)^k, \varphi(g)\sigma_g(z)^k) &= \text{dist}_{S_2}(\varphi(z^k), \varphi(gz^k)) \\ &= \text{dist}_{S_1}(z^k, gz^k) \\ &= \text{dist}_{S_1}(e, g), \end{aligned}$$

since $z \in Z(G_1)$. Then, from Lemma 2.7(10) (and the fact that Step 2 tells us that $\sigma_e(z)$ is in Z_2^\dagger and therefore commutes with $\varphi(g)$), we see that $\sigma_g(z) = \sigma_e(z)$. This means $\sigma_g(z)$ is independent of g (so we may drop the subscript).

Fix some $g \in G_1$ and $s \in S_1$. We have $\varphi(gs) = \varphi(g)\varphi(s)\sigma(z)$, for some $z \in Z_1^\dagger$ (because $\overline{\varphi}$ is a homomorphism and the surjectivity in Step 2 tells us $\sigma(Z_1^\dagger) = Z_2^\dagger$). Consider any $k \geq 0$ with $sz^k \in S_1$. Then

$$\varphi(gsz^k) = \varphi(gs)\sigma(z)^k = \varphi(g)\varphi(s)\sigma(z)\sigma(z)^k = \varphi(g)\varphi(sz^{k+1}).$$

Since φ is a graph homomorphism and, by assumption, $sz^k \in S_1$, we must have $\varphi(sz^{k+1}) \in S_2$. So $sz^{k+1} \in S_1$. By induction (with $k = 0$ as the base case), we conclude that $sz^k \in S_1$ for all $k \in \mathbb{Z}^+$. Since S_1 is finite (and G_1 is torsion-free), this implies $z = e$. So $\varphi(gs) = \varphi(g)\varphi(s)$. Since g is an arbitrary element of G_1 and s is an arbitrary element of the generating set S_1 , this implies that φ is a group homomorphism. \square

4. Nilpotent groups that may have torsion

Proposition 4.1. *Assume*

- S is a finite generating set of the group G , and
- N is a finite, normal subgroup of G , such that G/N is bi-orderable.

Then every automorphism of $\text{Cay}(G; S)$ induces a well-defined automorphism of $\text{Cay}(G/N; S)$.

Proof. Let

$$N^* = \{ \varphi(n) \mid \varphi \in \text{Aut}_e(\text{Cay}(G; S)), n \in N \}.$$

It is important to note that, since N is contained in a ball of finite radius centred at e , and N^* must be contained in that same ball, the set N^* is finite. We wish to show $N^* \subseteq N$.

Assume, without loss of generality, that $N \subseteq S$ (by passing to a power of $\text{Cay}(G; S)$). Since $\langle N^* \rangle$ is obviously invariant under $\text{Aut}_e(\text{Cay}(G; S))$, there is no harm in assuming $\langle N^* \rangle = G$.

Let $\overline{G} = G/N$, and let $\overline{N^*} = \{gN \mid g \in N^*\}$. We wish to show \overline{G} is trivial. Suppose not. (This will lead to a contradiction.) Since, by assumption, \overline{G} is bi-orderable, Lemma 2.11 provides $g \in N^*$, such that $[\overline{g}^i]_{i=-\infty}^{\infty}$ is a geodesic line in $\text{Cay}(\overline{G}; \overline{N^*})$. Then, since the natural map $\text{Cay}(G; N^*) \rightarrow \text{Cay}(\overline{G}; \overline{N^*})$ decreases distances, it is clear that $\gamma = [g^i]_{i=-\infty}^{\infty}$ is a geodesic line in $\text{Cay}(G; N^*)$. By the definition of N^* , there exists $\varphi \in \text{Aut}_e(\text{Cay}(G; S))$, such that $\varphi(g) \in N$. Then $\varphi(\gamma)$ is a geodesic line that contains the edge $e - n$ for some $n \in N$.

To obtain the contradiction that completes the proof, we use an argument of C. Löh [8, first paragraph of page 105]. Write $\varphi(\gamma) = [h_i]_{i=-\infty}^{\infty}$. For each $k \in \mathbb{N}$, let $\#(k)$ be the number of geodesic segments from h_i to h_{i+k} . (Since $\gamma = [g^i]_{i=-\infty}^{\infty}$ is obviously homogeneous, we know that $\varphi(\gamma)$ is also homogeneous, so $\#(k)$ is independent of the choice of i .) We may assume $h_0 = e$ (so $h_1 = n$). Since N is a finite normal subgroup of G , it is easy to see that no geodesic segment can contain two edges that are labelled by elements of N . (Namely, if (n, s_1, \dots, s_k, n') is a path in $\text{Cay}(G; N)$, then there exists $n'' \in N_1$, such that $n''s_1 \cdots s_k = ns_1 \cdots s_k n'$, so (n'', s_1, \dots, s_k) is a shorter path with the same endpoints.) Hence, for all $k > 1$, no geodesic segment from h_1 to h_k has any edges that are labelled by elements of N . (Otherwise, concatenating (n) at the start would yield a geodesic segment from h_0 to h_k with more than one edge labelled by elements of N .)

(see
note
A.19)

For any geodesic segment $\gamma' = (s_1, \dots, s_k)$ from h_1 to h_{k+1} , we can construct $k + 1$ different geodesic segments $\gamma_1, \dots, \gamma_{k+1}$ from h_0 to h_{k+1} , by inserting a single edge labelled by an element of N , as follows:

$$\gamma_i = (s_1, s_2, \dots, s_{i-1}, n_i, s_i, \dots, s_k),$$

where $n_i \in N$ is chosen so that $ns_1s_2 \cdots s_{i-1} = s_1s_2 \cdots s_{i-1}n_i$. (This is possible because the subgroup N is normal.) This implies $\#(k+1) \geq (k+1) \cdot \#(k)$, for all k . Therefore $\#(k) \geq k!$. However, it is clear that $\#(k) \leq |S|^k$, so this contradicts the fact that factorials grow faster than exponentials. \square

Combining this proposition with Theorem 1.3 yields the following slight generalization of Theorem 1.11 that allows G_1 and G_2 to be slightly non-nilpotent:

Theorem 4.2. *Assume*

- S_i is a symmetric, finite generating set of the group G_i , for $i = 1, 2$,
- N_i is a finite, normal subgroup of G_i , such that G_i/N_i is torsion-free nilpotent, for $i = 1, 2$, and
- φ is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$.

Then φ induces a well-defined affine bijection $\bar{\varphi}: G_1/N_1 \rightarrow G_2/N_2$.

Proof. By using φ to identify $\text{Cay}(G_1; S_1)$ with $\text{Cay}(G_2; S_2)$, we can realize G_2 as a sharply transitive subgroup G'_2 of $\text{Aut}(\text{Cay}(G_1; S_1))$. (Namely, for $h \in G_2$, define $h'(x) = \varphi^{-1}(h\varphi(x))$.)

For any $g \in G_1$ and $n \in N_1$, there exists $h \in G_2$, such that $\varphi(gn) = h\varphi(g)$. This means $h'g = gn \in gN_1$. From Proposition 4.1 (and Lemma 2.10), we know that G'_2 factors through to a well-defined group of permutations on G_1/N_1 , so this implies $h'(gN_1) = gN_1$. Since gN_1 is finite (and G'_2 is sharply transitive), we conclude that h' has finite order, so h' is in the torsion subgroup N'_2 of G'_2 . This means $h \in N_2$, so $\varphi(gn) = h\varphi(g) \in N_2\varphi(g)$. Therefore $\varphi(gN_1) \subseteq N_2\varphi(g)$. So φ induces a well-defined function $\bar{\varphi}: G_1/N_1 \rightarrow G_2/N_2$. \square

Definition 4.3 ([7, p. 305]). The *wreath product* (or *lexicographic product*) of two graphs $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ is the graph $X_1[X_2]$ with vertex set $V_1 \times V_2$, such that (v_1, v_2) is adjacent to (v'_1, v'_2) if and only if either

- v_1 is adjacent to v'_1 in X_1 , or
- $v_1 = v'_1$ and v_2 is adjacent to v'_2 in X_2 .

Proof of Corollary 1.12. (\Rightarrow) Let S_1 and S_2 be finite, symmetric generating sets of G_1 and G_2 , respectively, such that there is an isomorphism φ from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$. From Theorem 1.11, we know that φ induces a well-defined affine bijection $\bar{\varphi}: G_1/N_1 \rightarrow G_2/N_2$. By composing

with a left-translation, we may assume $\overline{\varphi}$ is a group isomorphism. Obviously, this implies $G_1/N_1 \cong G_2/N_2$. Also, since $\overline{\varphi}$ is a well-defined bijection, we must have $\varphi(N_1) = N_2$. Since φ is a bijection, this implies $|N_1| = |N_2|$.

(\Leftarrow) Let

- $\overline{\varphi}$ be an isomorphism from G_1/N_1 to G_2/N_2 ,
- $\overline{S_1}$ be a finite generating set of G_1/N_1 , with $e \notin \overline{S_1}$,
- $\overline{S_2} = \overline{\varphi}(\overline{S_1})$ be the corresponding generating set of G_2/N_2 , and
- $S_i = \{s \in G_i \mid sN_i \in \overline{S_i}\}$, for $i = 1, 2$.

Let $n = |N_1| = |N_2|$, and let E_n be the edgeless graph on n vertices. Then, for $i = 1, 2$, it is easy to see that $\text{Cay}(G_i; S_i)$ is isomorphic to the wreath product $\text{Cay}(G_i/N_i; \overline{S_i})[E_n]$. Since it is obvious that $\overline{\varphi}$ is an isomorphism from $\text{Cay}(G_1/N_1; \overline{S_1})$ to $\text{Cay}(G_2/N_2; \overline{S_2})$, we have $\text{Cay}(G_1; S_1) \cong \text{Cay}(G_2; S_2)$. \square

Proof of Corollary 1.13. Let H be a sharply transitive, nilpotent subgroup of $\text{Aut}(\text{Cay}(G; S))$. Then a well-known result of G. Sabidussi tells us that $\text{Cay}(G; S)$ is isomorphic to a Cayley graph on H [7, Prop. 1.1], so Corollary 1.12 implies $G \cong H$. (see
note
A.20)

From Theorem 1.3, we see that if S' is any symmetric, finite subset of G , such that $\text{Cay}(G; S') \cong \text{Cay}(G; S)$, then there is a group automorphism α of G with $\alpha(S) = S'$. Therefore, since H is a sharply transitive subgroup of $\text{Aut}(\text{Cay}(G; S))$ that is isomorphic to G , a well-known theorem of L. Babai tells us that H is conjugate in $\text{Aut}(\text{Cay}(G; S))$ to the left-regular representation of G [7, Thm. 4.1]. However, Corollary 1.7 states that the left-regular representation has no other conjugates in $\text{Aut}(\text{Cay}(G; S))$, so we conclude that H is equal to the left-regular representation of G . \square (see
note
A.21)

5. Other groups that have torsion

In this section, we prove Proposition 1.9. In fact, we prove a more specific version of Proposition 1.9:

Proposition 5.1. *Suppose F is a nontrivial, finite subgroup of a group G , and S is any finite, symmetric generating set for G . Then $\text{Cay}(G; FSF)$ is a connected Cayley graph of finite valency that is not normal.*

Proof. It is straightforward to verify that FSF is a symmetric, finite generating set of G , so $\text{Cay}(G; FSF)$ is a connected Cayley graph of finite valency. Furthermore, for all $g \in G$, it is straightforward to check that all vertices in the coset gF have the same neighbours. Therefore, if we choose some $h \in gF$ (with $h \neq g$), then there is an automorphism φ of $\text{Cay}(G; FSF)$ that interchanges g and h , but fixes all other vertices of the Cayley graph. Since G is infinite, but FSF is finite, we may assume g has been chosen so that gF is disjoint from $FSF \cup \{e\}$. Then φ fixes e , but is obviously not a group automorphism, since it fixes every element of the generating set FSF , (see
note
A.23
(see
note
A.24)

and is not the identity map (since it moves g to h). So φ is not an affine bijection. \square

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Appendix A. Notes to aid the referee

A.1. See Remark 1.14(4) for an example of isomorphic Cayley graphs on non-isomorphic groups.

Definition A.1.1. Let S be a subset of a group G .

- S is *symmetric* if it is closed under inverses; that is, $s^{-1} \in S$ for all $s \in S$.
- If S is symmetric, then the corresponding *Cayley graph* on G is the graph $\text{Cay}(G; S)$ whose vertices are the elements of G , and with an edge $g \text{ --- } gs$, for all $g \in G$ and $s \in S$.

Remark A.1.2. It is easy to see that $\text{Cay}(G; S)$ is connected if and only if S generates G .

A.2. We show that Theorem 1.3 implies Theorem 1.1. Let φ be an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$. From Theorem 1.3, we know there exist a group isomorphism $\alpha: G_1 \rightarrow G_2$ and $h \in G_2$, such that $\varphi(x) = h \cdot \alpha(x)$, for all $x \in G_1$. Since α is a group isomorphism, we have $G_1 \cong G_2$.

A.3. The *left-regular representation* of G is the set $\{\hat{g} \mid g \in G\}$ of permutations of G , where $\hat{g}: G \rightarrow G$ is defined by $\hat{g}(x) = gx$ for $x \in G$. Since $\widehat{gh} = \hat{g}\hat{h}$, this is a subgroup of the symmetric group on the set G .

A.4. It is clear that φ is a bijection. The neighbours of $a^i b^j$ are $a^{i\pm 1} b^j$ and $a^i b^{j\pm 1}$. These neighbours are mapped by φ to

$$b^{i\pm 1} a^j = b^i a^j \cdot b^{\pm 1} = \varphi(a^i b^j) b^{\pm 1}$$

and

$$b^i a^{j\pm 1} = b^i a^j \cdot a^{\pm 1} = \varphi(a^i b^j) a^{\pm 1},$$

which are neighbours of $\varphi(a^i b^j)$. So φ is a graph automorphism.

Suppose φ is an affine bijection. Since $\varphi(e) = e$ (and φ is a bijection), this implies that φ is an automorphism of the group G . However, we have $\varphi(a) = b$, and no automorphism of G can map a to b , since $\langle a \rangle \triangleleft G$, but $\langle b \rangle \not\triangleleft G$. This is a contradiction.

A.5. Proposition 2.1 follows from the following weaker conclusion that does not require the assumption that S_i generates G_i .

Lemma A.5.1. *Assume*

- G_1 and G_2 are torsion-free, abelian groups,
- S_i is a symmetric, finite subset of G_i , for $i = 1, 2$, and
- φ is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$.

Then, for each $g \in G_1$ and $s \in S_1$, there exists $\sigma_g(s) \in S_2$, such that $\varphi(gs^k) = \varphi(g)\sigma_g(s)^k$ for all $k \in \mathbb{Z}$.

Proof. To simplify the notation, assume $\langle S_i \rangle = G_i$ for $i = 1, 2$. (This causes no loss of generality, since $\varphi(g\langle S_1 \rangle) = \varphi(g)\langle S_2 \rangle$ for all $g \in G$, but a detailed proof works with cosets of $\langle S_i \rangle$, instead of the subgroup $\langle S_i \rangle$ itself.) Lemma 2.11 provides $s_1 \in S_1$, such that $[s_1^k]_{k=-\infty}^{\infty}$ is a convex geodesic line in $\text{Cay}(G_1; S_1)$. Then $[gs_1^k]_{k=-\infty}^{\infty}$ is also a convex geodesic line in $\text{Cay}(G_1; S_1)$ (since left-translation is an automorphism of the Cayley graph). Applying the isomorphism φ yields the convex geodesic line $[\varphi(gs_1^k)]_{k=-\infty}^{\infty}$ in $\text{Cay}(G_2; S_2)$. Now Lemma 2.4 implies that all edges in this geodesic line have the same label (since G_2 is abelian). This means there is some $\sigma_g(s_1) \in S_2$, such that $\varphi(gs_1^k) = \varphi(g)\sigma_g(s_1)^k$ for all $k \in \mathbb{Z}$. This is the desired conclusion for $s = s_1$.

Now, we make the important observation that if $\sigma_g(s)$ exists, for some $s \in S_1$, then $\sigma_g(s) = \sigma_h(s)$ for all $g, h \in \langle S_1 \rangle$. Namely, for all $k \in \mathbb{Z}$, we have

$$\begin{aligned} \text{dist}_{S_1}(g, h) &= \text{dist}_{S_1}(gs^k, hs^k) && (G_1 \text{ is abelian}) \\ &= \text{dist}_{S_2}(\varphi(gs^k), \varphi(hs^k)) && (\varphi \text{ is an isomorphism}) \\ &= \text{dist}_{S_2}(\varphi(g)\sigma_g(s)^k, \varphi(h)\sigma_h(s)^k), \end{aligned}$$

so Lemma 2.7(10) tells us that $\sigma_g(s) = \sigma_h(s)$.

Therefore, $\sigma_g(s_1)$ is a constant (since we assumed at the start of the proof that $\langle S_1 \rangle = G_1$; without this assumption, it would only be constant on cosets of $\langle S_1 \rangle$). Calling this constant s_2 yields $\varphi(gs_1) = \varphi(g)s_2$ for all $g \in G_1$. Letting $S'_i = S_i \setminus \{s_i^{\pm 1}\}$ for $i = 1, 2$, this implies that φ is an isomorphism from $\text{Cay}(G_1; S'_1)$ to $\text{Cay}(G_2; S'_2)$. By induction on the valency, we conclude that the desired $\sigma_g(s)$ exists for all $s \in S_1 \setminus \{s_1^{\pm 1}\}$. Since the first paragraph provides $\sigma_g(s_1)$, this completes the proof. \square

Proof of Proposition 2.1. Since $\langle S_1 \rangle = G_1$, the second paragraph of the proof of the lemma tells us that $\sigma_g(s) = \sigma_h(s)$ for all $g, h \in G_1$, so we may drop the subscript: $\varphi(gs) = \varphi(g)\sigma(s)$ for all $g \in G_1$ and $s \in S_1$. Since S_1 generates G_1 , and φ is a bijection, this implies that φ is an affine bijection. \square

A.6. Let $g \in G$ and $x \in \sqrt{N}$. There is some $k > 0$ with $x^k \in N$. Since $N \trianglelefteq G$, we have

$$(g^{-1}xg)^k = g^{-1}x^kg \in g^{-1}Ng = N,$$

so $g^{-1}xg \in \sqrt{N}$. Therefore $\sqrt{N} \trianglelefteq G$.

Suppose $g\sqrt{N}$ is a torsion element of G/\sqrt{N} . This means there is some $k \neq 0$ with $(g\sqrt{N})^k = \sqrt{N}$, so $g^k \in \sqrt{N}$. This means there is some $\ell \neq 0$ with $(g^k)^\ell \in N$. Therefore $g^{k\ell} \in N$ (and $k\ell \neq 0$), so $g \in \sqrt{N}$. Therefore $g\sqrt{N}$ is trivial. So G/\sqrt{N} is torsion-free.

A.7.

Lemma A.7.1 ([4, Lem. 2.6, p. 9]). *Let G be nilpotent of class c and let H be a proper subgroup of G . Define $H_0 = H$ and, inductively, H_{i+1} to be the normalizer of H_i in G . Then*

$$H = H_0 < H_1 < \cdots < H_r = G$$

for some $r \leq c$.

Proof of Lemma 2.7(7). Let H_i be as in Lemma A.7.1. Since H_{i+1} is the normalizer of H_i , we may write

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_r = G.$$

From Lemma 2.7(6) and induction, we have

$$\text{rank } G = \text{rank } H + \sum_{i=1}^r \text{rank}(H_i/H_{i-1}).$$

So $\text{rank } H \leq \text{rank } G$, with equality if and only if $\text{rank}(H_i/H_{i-1}) = 0$ for all i .

Since it is clear that $\text{rank } F = 0$ if and only if F is finite, this means that $\text{rank } H = \text{rank } G$ if and only if H_i/H_{i-1} is finite for all i . This is the case if and only if G/H is finite. \square

A.8. If we take the special case of π -isolated where π is the set of all prime numbers, [6, 2.3.8(i), p. 42] says:

Suppose H is a subgroup of a torsion-free, nilpotent group G .
Then $C_G(H)$ is isolated for every subgroup H .

To say that $C_G(H)$ is “isolated” means that if $g^k \in C_G(H)$ for some nonzero $k \in \mathbb{Z}$, then $g \in C_G(H)$ [6, first paragraph of §2.3, p. 38].

Now, suppose h has only finitely many conjugates. This means $C_G(h)$ is a finite-index subgroup of G , so there is some nonzero $k \in \mathbb{Z}$, such that $g^k \in C_G(h)$ for all $g \in G$. From the preceding paragraph, we conclude that $g \in C_G(h)$. Since g is an arbitrary element of G , this means $h \in Z(G)$.

A.9. Since $\text{dist}_S(a^k, gb^k)$ is bounded as a function of k , we know that

$$\{a^{-k}gb^k \mid k \in \mathbb{Z}\} \text{ is finite.}$$

Hence, there exist $k \neq \ell$, such that $a^{-k}gb^k = a^{-\ell}gb^\ell$, so, letting $m = \ell - k \neq 0$, we have $g^{-1}a^m g = b^m$. In other words, $(g^{-1}ag)^m = b^m$. Since G is torsion-free nilpotent, this implies $g^{-1}ag = b$ [6, 2.1.2, p. 30].

A.10. The paper [1] uses the following notation:

- [1, Defns. 2.2 and 2.3] $\|x\| = \text{dist}_S(e, x)$ (this is called a “word metric”)
- [1, Lem. 2.43(i)] $\tau(x) = \lim_{n \rightarrow \infty} \|x^n\|/n$
- [1, Defn. 2.5] $I(G) = \{g \in G \mid \tau(g) = 0\}$
- [1, Defn. 3.1] $B(G) = \{g \in G \mid \tau(gx) = \tau(x), \forall x \in G\}$.
- [1, Notn. 3.2(ii)] $G' = [G, G]$

Lemma A.10.1 ([1, Lem. 3.5(i,iii)]). *Let G be a nilpotent group. Then*

- (i) $B(G) = I(G)$
- (iii) *If G is finitely generated and equipped with a word metric then $B(G) = \sqrt{G'}$.*

Proof of Lemma 2.7(11). Translating to the notation of [1], we have

$$\text{dist}_S(e, g^k)/k \rightarrow 0 \Leftrightarrow \|g^k\|/k \rightarrow 0 \Leftrightarrow \tau(g) = 0 \Leftrightarrow g \in I(G).$$

From Lemma A.10.1, we have $I(G) = B(G) = \sqrt{G'} = \sqrt{[G, G]}$. □

A.11. We have

$$x \prec' y \Rightarrow x^{-1} \prec y^{-1} \Rightarrow b^{-1}x^{-1}a^{-1} \prec b^{-1}y^{-1}a^{-1} \Rightarrow axb \prec' ayb,$$

so \prec' is invariant under both left-translations and right-translations.

Also, from the definition of \prec' , we have $s \succeq (S \cup S^{-1}) \Leftrightarrow s^{-1} \succeq' (S \cup S^{-1})$.

A.12. Since $a \preceq b$, invariance under right-translations implies $ac \preceq bc$ (with equality iff $a = b$). Since $c \preceq d$, invariance under left-translations implies $bc \preceq bd$ (with equality iff $c = d$). Now transitivity implies $ac \preceq bd$ (with equality iff $a = b$ and $c = d$).

For the base case of a proof by induction, note that the maximality of s implies $s_1 \preceq s$ (with equality iff $s_1 = s$). Now suppose $s_1s_2 \cdots s_k \preceq s^k$ (with equality iff $s_1 = s_2 = \cdots = s_k = s$). Since $s_1s_2 \cdots s_k \preceq s^k$ and $s_{k+1} \preceq s$, we have

$$s_1s_2 \cdots s_{k+1} = s_1s_2 \cdots s_k \cdot s_{k+1} \preceq s^k \cdot s = s^{k+1},$$

with equality iff $s_1s_2 \cdots s_k = s^k$ and $s_{k+1} = s$. However, we have already noted that $s_1s_2 \cdots s_k = s^k$ implies $s_1 = s_2 = \cdots = s_k = s$.

A.13. Let $h = \varphi(e)$, and define $\varphi'(x) = h^{-1} \cdot \varphi(x)$. Then φ' is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$ with $\varphi'(e) = e$. If φ' is an affine bijection, then φ is also an affine bijection.

A.14. Let $h = \varphi(g)$, and define $\varphi'(x) = h^{-1} \cdot \varphi(gx)$. Then φ' is an isomorphism from $\text{Cay}(G_1; S_1)$ to $\text{Cay}(G_2; S_2)$ with $\varphi'(e) = e$. If there is some $g' \in G_2$, such that $\varphi'(z^k) = \varphi'(e)(g')^k$, for all $k \in \mathbb{Z}$, then

$$\varphi(gz^k) = h \cdot \varphi'(z^k) = \varphi(g) \cdot \varphi'(e)(g')^k = \varphi(g)(g')^k,$$

so we may let $\sigma_g(z) = g'$.

A.15. The definition of S_2^* provides $\rho \in \text{Aut}_e(\text{Cay}(G_2; S_2))$ with $\rho(g_2) = s$. Since $g_2 = \varphi(z)$, we may let ψ be the composition $\rho \circ \varphi$.

A.16. Suppose G_1^* has finite index in G_1 . Then $\sqrt{G_1^*} = G_1$, so Lemma 2.7(5) implies

$$[G_1, G_1] = [\sqrt{G_1^*}, \sqrt{G_1^*}] \subseteq \sqrt{[G_1^*, G_1^*]},$$

so $[G_1^*, G_1^*]$ has finite index in $[G_1, G_1]$ (see Lemma 2.7(2)). This is a contradiction.

A.17. For a graph Γ and $r \in \mathbb{Z}^+$, the r th power of Γ is the graph Γ^r with the same vertex set as Γ , and with an edge from u to v iff $\text{dist}_\Gamma(u, v) \leq r$. It is clear that:

- Any isomorphism from Γ_1 to Γ_2 is also an isomorphism from Γ_1^r to Γ_2^r .
- $\text{Cay}(G; S)^r = \text{Cay}(G; S^r)$, where S^r is the set of all elements of G that can be written as a product of $\leq r$ elements of S .

Since Z_1^\dagger is finitely generated, it has a finite generating set. For any sufficiently large r , this finite set is contained in S_1^r . Since φ is an isomorphism from $\text{Cay}(G_1; S_1^r)$ to $\text{Cay}(G_2; S_2^r)$, there is no harm in replacing S_1 and S_2 with S_1^r and S_2^r .

A.18. We have

$$\begin{aligned} \sqrt{Z_i^\dagger} &= \sqrt{Z(G_i) \cap \sqrt{[G_i, G_i]}} && \text{(definition of } Z_i^\dagger) \\ &= \sqrt{Z(G_i)} \cap \sqrt{\sqrt{[G_i, G_i]}} && (\sqrt{H \cap K} = \sqrt{H} \cap \sqrt{K}) \\ &= Z(G_i) \cap \sqrt{[G_i, G_i]} && \text{(Lemma 2.7(4) and } \sqrt{\sqrt{H}} = \sqrt{H}) \\ &= Z_i^\dagger && \text{(definition of } Z_i^\dagger), \end{aligned}$$

so G/Z_i^\dagger is torsion-free.

A.19. Suppose we can show that the result is true for $\langle N^* \rangle$. Let f be an automorphism of $\text{Cay}(G; S)$ that fixes e , and let f^* be the restriction of f to $\langle N^* \rangle$. Since N^* is invariant, we know that f^* is an automorphism of $\text{Cay}(\langle N^* \rangle; N^*)$. Also, it is clear from the definition of N^* that N is contained in N^* . (Also, $\langle N^* \rangle/N$ is bi-orderable, because it is a subgroup of G/N .) Therefore, if we know the theorem is true for $\langle N^* \rangle$, then $f^*(N)$ is contained in N . Since $f^*(N) = f(N)$, this means that $f(N)$ is contained in N , as desired.

A.20.

Proposition A.20.1 (Sabidussi, 1964). *A graph Γ is isomorphic to a Cayley graph on a group G if and only if $\text{Aut } \Gamma$ contains a sharply transitive subgroup that is isomorphic to G .*

Now, let N be the torsion subgroup of H . Since G and H both have a Cayley graph isomorphic to $\text{Cay}(G; S)$ (and the torsion subgroup of G is trivial), Corollary 1.12 tells us that $G/\{e\} \cong H/N$ and $|\{e\}| = |N|$. So $G \cong H$.

A.21. Let φ be an isomorphism from $\text{Cay}(G; S)$ to $\text{Cay}(G; S')$. From Theorem 1.3, we know that φ is an affine bijection, so there exist a group automorphism α of G and $h \in G$, such that $\varphi(x) = h \cdot \alpha(x)$ for all $x \in G$. Since φ is a graph isomorphism, we have $\varphi(xS) = \varphi(x)S'$ for all $x \in S$. Taking $x = e$, this yields

$$h \cdot \alpha(S) = \varphi(eS) = \varphi(e)S' = h \cdot \alpha(e)S' = h \cdot S',$$

so $\alpha(S) = S'$.

A.22. The following result is traditionally stated only for finite groups, but the same proof works in general.

Proposition A.22.1 (Babai, 1977). *For a group G , the following two conditions are equivalent:*

- *whenever S and S' are finite, symmetric generating sets of G and $\text{Cay}(G; S) \cong \text{Cay}(G; S')$, there is an automorphism α of G with $\alpha(S) = S'$;*
- *for every finite, symmetric generating set S of G , the left-regular representation of G is conjugate to every subgroup of $\text{Aut}(\text{Cay}(G; S))$ that is isomorphic to G and acts sharply transitively on the vertices of $\text{Cay}(G; S)$.*

A.23. We have $(FSF)^{-1} = F^{-1}S^{-1}F^{-1} = FSF$ (since F and S are symmetric), so FSF is symmetric. Also, it is clear that FSF is finite, since F and S are both finite. Finally, since $e \in F$ (because F is a subgroup), we have $S = e \cdot S \cdot e \subseteq FSF$, so FSF generates G .

A.24. For $f \in F$, the set of neighbours of gf is $gf \cdot FSF = g \cdot (fF) \cdot SF = gFSF$, which is the set of neighbours of g .